PAUTA C1 EDO 2021

(E) $\quad (1-t^2)x''(t) - 2t x'(t) + n(n+1)x(t) = 0 \;, \quad t\in(-1,1)$, $n\in\mathbb{N}$.

Parte 1:

(2pts) $\boxed{n=0}$ $\quad (1-t^2)x''(t) - 2tx'(t) = 0 \;, \; t\in(-1,1).$

Sea $y(t) := x'(t)$, de donde $(1-t^2)y'(t) - 2t\,y(t) = 0.$ (0.5pts)

Luego, $y'(t) - \frac{2t}{1-t^2}y(t) = 0.$ (EDO orden 1 lineal)

Factor integrante: $\mu(t) := e^{\int -2t/1-t^2} = e^{\int \frac{d}{dt}\ln(1-t^2)}$

$= e^{\ln(1-t^2)} = 1-t^2.$ (0.5 pts)

Multiplicando la EDO por este factor integrante,

$(1-t^2)\,y'(t) - 2t\,y(t) = 0 \iff ((1-t^2)y(t))' = 0.$

$\Rightarrow \boxed{y(t) = \frac{c_1}{1-t^2}} \Rightarrow x(t) = \int \frac{c_1}{1-t^2} + c_2 \underset{\text{dato}}{=} c_2 + \frac{c_1}{2}\ln\left(\frac{1+t}{1-t}\right)$

Conclusión: $\boxed{x(t) = \frac{c_1}{2}\ln\left(\frac{1+t}{1-t}\right) + c_2}$ (0.5 pts)

Si ahora $x(0)=1$, $x'(0)=0$, $\quad x(0) = 1 = \frac{c_1}{2}\overset{0}{\cancel{\ln 1}} + c_2$

$\Rightarrow \boxed{c_2 = 1}$

$x'(t)\Big|_{t=0} = 0 = \frac{c_1}{1-t^2}\Big|_{t=0} = c_1 \Rightarrow \boxed{c_1 = 0}$

Luego, la solución es $\boxed{x(t) \equiv 1}$ (0.5 pts)

2. $\underline{n=0}$ $(1-t^2)x''(t) - 2t\,x'(t) = f(t) = \dfrac{1}{1-t^2}$.

$\underline{\text{Soluciones}}$: $x_1(t) = 1$, $x_2(t) = \dfrac{1}{2}\ln\left(\dfrac{1+t}{1-t}\right)$. $\boxed{0.5 \text{ pts}}$

Veamos el Wronskiano:

$$W = W(x_1, x_2) = \begin{vmatrix} 1 & \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) \\ 0 & \frac{1}{1-t^2} \end{vmatrix} = \frac{1}{1-t^2}. \quad (\neq 0, \text{ ie., } x_1, x_2 \text{ son li})$$

$\boxed{0.5 \text{ pts}}$

$\underline{\text{Variación de parámetros}}$:

$$x_p(t) = -x_1 \int \frac{x_2 f}{W} + x_2 \int \frac{x_1 f}{W}$$

$$= -1 \int \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) \frac{1}{\cancel{1-t^2}} \cancel{(1-t^2)} + \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) \int \frac{1}{1-t^2}$$

(IPP)

$$= -t \cdot \frac{1}{2}\cancel{\ln\left(\frac{1+t}{1-t}\right)} + \int \frac{t}{1-t^2} + \frac{1}{2}\cancel{\ln\left(\frac{1+t}{1-t}\right)}\, t$$

$\boxed{0.5 \text{ pts}}$

$$= -\frac{1}{2}\ln(1-t^2)$$

Luego, $\boxed{x(t) = C_1 1 + \dfrac{C_2}{2}\ln\left(\dfrac{1+t}{1-t}\right) - \dfrac{1}{2}\ln(1-t^2)}$

$\boxed{0.5 \text{ pts}}$

3. $\boxed{n=1}$ $(1-t^2)\,x''(t) - 2t\,x'(t) + 2\,x(t) = 0.$

$$\left.\begin{aligned} x(t) &= a + bt \\ x'(t) &= b \\ x''(t) &= 0 \end{aligned}\right\} \quad (1-t^2)\cdot 0 - 2tb + 2(a+bt) = 0$$

$$\Rightarrow 2a = 0 \Rightarrow \boxed{a=0}. \qquad \boxed{0.5 \text{ pts}}$$

Luego, $x_1(t) = t$ es una solución.

Para la otra, usamos <u>reducción de orden</u> en

$$x''(t) - \frac{2t}{1-t^2}x'(t) + \frac{2}{1-t^2}x(t) = 0.$$

$$x_2(t) = x_1(t)\int \frac{e^{-\int\left(-2t/1-t^2\right)}}{x_1^2} = t\int \frac{e^{\int\frac{2t}{1-t^2}}}{t^2}$$

$$= t\int \frac{e^{-\ln(1-t^2)}}{t^2} = t\int \frac{1}{t^2(1-t^2)} = \left(-\frac{1}{t} + \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)\right)$$

dato

$\boxed{1\text{pto}}$ $$\boxed{x_2(t) = -1 + \frac{1}{2}t\ln\left(\frac{1+t}{1-t}\right)}$$

<u>Solución completa</u>: $$\boxed{x(t) = c_1 t + c_2\left(-1 + \frac{1}{2}t\ln\left(\frac{1+t}{1-t}\right)\right)}$$

$\boxed{0.5 \text{ pts}}$

Parte 2 | 1. $\underline{n=2}$ $\quad (1-t^2)x''(t) - 2t x'(t) + 6x(t) = 0.$

$$x(t) = a + bt + ct^2 \Rightarrow \begin{cases} x'(t) = b + 2ct \\ x''(t) = 2c. \end{cases} \quad \boxed{0.5 \text{ pts}}$$

Reemplazando,

$$\begin{aligned} 0 &= (1-t^2)2c - 2t(b+2ct) + 6(a+bt+ct^2) \\ &= (2c+6a) - 2bt + 6bt + (-2c - 4c + 6c)t^2 \\ &= (6a+2c) + 4bt \Rightarrow \boxed{b=0, \quad c=-3a} \leftarrow \boxed{1 \text{pto}} \end{aligned}$$

$$\Rightarrow \boxed{x(t) = a - 3at^2 = a(1-3t^2)}, \quad a \in \mathbb{R} \text{ libre.}$$

Si queremos $x(\pm 1) = 1 \Rightarrow 1 = a(1-3) = -2a$

$$\Rightarrow \boxed{a = -\frac{1}{2}}.$$

$$\Rightarrow \boxed{x(t) = -\frac{1}{2}(1-3t^2)}$$ [redacted] $\boxed{1 \text{pto}}$

2. Sea $y(t) = x'(t)$. Entonces (E) se escribe

$$\left.\begin{cases} x'(t) = y(t) \\ y'(t) = x''(t) = \dfrac{2t}{1-t^2}x'(t) - \dfrac{n(n+1)}{1-t^2}x(t) \\ \quad\quad = \dfrac{2t}{1-t^2}y(t) - \dfrac{n(n+1)}{1-t^2}x(t). \end{cases}\right\} \boxed{1 \text{pto}}$$

Luego, tenemos el sistema

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) := y(t) \\ y'(t) = f_2(t, x(t), y(t)) := \dfrac{2t}{1-t^2}y(t) - \dfrac{n(n+1)}{1-t^2}x(t) \end{cases}$$

Ahora invocamos predictor-corrector:

- Dividimos $[0, \frac{1}{2}]$ en N intervalos de largo h (excepto el último tal vez),

$$\left.\begin{aligned} & t_i = ih, \quad i = 0, \dots, N \\ & \begin{cases} x_0 = x_0 \text{ fijo dado} \\ y_0 = y_0 \quad '' \quad ''. \end{cases} \end{aligned}\right\} \boxed{0.5 \text{ pts}}$$

$\forall i = 0, 1, 2, \dots$

$\boxed{1 pto}$

predictor $\begin{cases} x^*_{i+1} = x_i + h f_1(t_i, x_i, y_i) = x_i + h y_i \\ y^*_{i+1} = y_i + h f_2(t_i, x_i, y_i) = y_i + h \left(\frac{2t_i}{1-t_i^2} y_i - \frac{n(n+1)}{1-t_i^2} x_i \right). \end{cases}$

corrector: $\begin{cases} x_{i+1} = x_i + \frac{h}{2} \left(f_1(t_i, x_i, y_i) + f_1(t_{i+1}, x^*_{i+1}, y^*_{i+1}) \right) \\ \qquad = x_i + \frac{h}{2} (y_i + y^*_{i+1}) \checkmark \\ y_{i+1} = y_i + \frac{h}{2} \left(f_2(t_i, x_i, y_i) + f_2(t_{i+1}, x^*_{i+1}, y^*_{i+1}) \right) \\ \qquad = y_i + \frac{h}{2} \left(\frac{2t_i}{1-t_i^2} y_i - \frac{n(n+1)}{1-t_i^2} x_i + \frac{2t_{i+1}}{1-t_{i+1}^2} y^*_{i+1} - \frac{n(n+1)}{1-t_{i+1}^2} x^*_{i+1} \right) \end{cases}$

$\boxed{1 pto}$